Journal of Global Optimization **17:** 65–75, 2000.

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Periodic Solutions of Superlinear Convex Autonomous Hamiltonian Systems*

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(Received for publication May 2000)

Abstract. Using the Szulkin's variant of Mountain Pass Theorem, we prove the existence of nontrivial orbits with prescribed period for autonomous Hamiltonian systems in infinite dimensional Hilbert spaces.

Key words: Nontrivial periodic solutions; Szulkin's variant of mountain pass theorem

AMS Subject Classification: 58E50, 34C25

1. Introduction

In Mawhin and Willem (1989), the authors combining the dual least action principle and the classical variant of mountain pass theorem, have proved the existence of nontrivial periodic solutions for the autonomous Hamiltonian system

$$J\dot{u}(t) + \nabla H(u(t)) = 0 \qquad (1)$$

where $H \in C^1(\mathbf{R}^{2N}, \mathbf{R})$ is strictly convex, $H(0) = 0, \nabla H(0) = 0$ and $J : \mathbf{R}^{2N} \to \mathbf{R}^{2N}$ is symplectic matrix. More specifically, Jean Mawhin and Michel Willem have proved the following result:

THEOREM (see Mawhin and Willem (1989), Theorem 4.11). *If there exists* $q > 2$, $\alpha > 0$, *such that, for every* $u \in \mathbf{R}^{2N}$, *one has*

$$qH(u) \leqslant \langle \nabla H(u), u \rangle$$

and

$$H(u) \leqslant \alpha \|u\|^q ,$$

then, for each $T > 0$, *(1) has a nontrivial T-periodic solution.*

This theorem was first proved by Rabinowitz (1978) by a direct minimax

* This paper is dedicated to the memory of Professor P.D. Panagiotopoulos.

approach and without the convexity assumption. The proof given in Mawhin and Willem (1989) is due to Ekeland (1979).

In this paper we will consider the following problem

$$\mathscr{A}u(t) = \nabla H(u(t)) , \tag{2}$$

where $A : D(A) \subset X \to X$ and $-A$ generates a C_0-semigroup e^{-At} on the infinite dimensional Hilbert space X and e^{-AT} is compact and

$$\mathscr{A} : D(\mathscr{A}) \subset \tilde{X} \to \tilde{X} , \qquad \mathscr{A}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} A^*p - \dot{p} \\ Aq + \dot{q} \end{pmatrix} . \tag{3}$$

If we will introduce the operator

$$\tilde{\mathscr{A}} : D(\tilde{\mathscr{A}}) \subset L^2(0, T; \tilde{X}) \to L^2(0, T; \tilde{X}) \tag{4}$$

where

$$D(\tilde{\mathscr{A}}) = \{u \in L^2(0, T; \tilde{X}) : u(0) = u(T),\ \dot{u} \in L^2(0, T; \tilde{X})\}$$

defined by

$$(\tilde{\mathscr{A}}u)(t) = \mathscr{A}u(t) , \quad \text{for all } t \in [0, T] ,$$

follows that

– $\tilde{\mathscr{A}}$ is linear, densely defined closed operator with closed range $R(\tilde{\mathscr{A}})$ (see Barbu (1996));

– $\tilde{\mathscr{A}}^{-1} : R(\tilde{\mathscr{A}}) \to L^2(0, T; \tilde{X})$ is linear and compact (see Barbu (1996)).

We will also assume that $H \in C^1(\tilde{X}, \mathbf{R})$ is uniformly convex such that $H(\theta_{\tilde{X}}) = 0$, $\nabla H(\theta_{\tilde{X}}) = \theta_{\tilde{X}}$ and exists
$\alpha > 0$ such that, for every $u \in \tilde{X}$, one has

$$2H(u) \leqslant \langle \nabla H(u), u \rangle \tag{5}$$

and

$$H(u) \leqslant \alpha \|u\|^2_{\tilde{X}} , \qquad \|\mathscr{A}^{-1}\| < \frac{1}{2\alpha} . \tag{6}$$

In this case, we can prove that the corresponding dual action defined by

$$\varphi(v) = \int_0^T H^*(v(t))\, dt - \frac{1}{2} \langle \tilde{\mathscr{A}}^{-1} v, v \rangle_{L^2(0,T;\tilde{X})}$$

is differentiable on $R(\tilde{\mathscr{A}})$. Because we do not know if the dual action is continuously differentiable, we cannot apply the classical variant of mountain pass theorem for obtaining a critical point for φ. This is the reason for which we will write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1(v) = -\frac{1}{2} \int_0^T \langle \mathscr{A}^{-1} v(t),\ v(t) \rangle_{\tilde{X}}\, dt$, $\varphi_2(v) = \int_0^T H^*(v(t))\, dt$, and we will prove that φ_1 is continuously differentiable and φ_2 is convex and differentiable. Using the Szulkin's variant of mountain pass theorem (see Szulkin (1986)), we find the critical point for φ.

2. The main result

THEOREM 1. *Let X be an infinite dimensional Hilbert space, $\tilde{X} = X \times X$, and $H \in C^1(\tilde{X}, \mathbf{R})$ is uniformly convex such that $H(\theta_{\tilde{X}}) = 0$, $\nabla H(\theta_{\tilde{X}}) = \theta_{\tilde{X}}$. If there exists $\alpha > 0$, such that, for every $u \in \tilde{X}$, one has*

$$2H(u) \leqslant \langle \nabla H(u), u \rangle \tag{7}$$

and

$$H(u) \leqslant \alpha \|u\|_{\tilde{X}}^2, \qquad \|\mathscr{A}^{-1}\| < \frac{1}{2\alpha} \tag{8}$$

then, for each $T > 0$

$$\mathscr{A}u(t) = \nabla H(u(t)) \tag{9}$$

has a nontrivial T-periodic solution.

Proving Theorem 1, requires several definitions and preliminary results.

3. The preliminary results

We introduce some functional spaces. Let $[0, T]$ be a fixed real interval $(0 < T < \infty)$ and let X be a Hilbert space. We introduce the Hilbert space $\tilde{X} = X \times X$ with the natural inner product

$$\langle u, v \rangle_{\tilde{X}} = \langle q_1, q_2 \rangle_X + \langle p_1, p_2 \rangle_X, \quad \forall u = (q_1, p_1), \quad v = (p_1, p_2) \in \tilde{X},$$

and we denote by $L^p(0, T; \tilde{X})$ the space of all (classes of) strongly measurable functions $u : [0, T] \to \tilde{X}$ such that $\int_0^T \|u(t)\|_{\tilde{X}}^p \, dt < \infty$ for $1 \leqslant p < \infty$. We denote by $W_T^{1,p}(\tilde{X}) = \{u : [0, T] \to \tilde{X} : u, \quad \dot{u} \in L^p(0, T; \tilde{X})\}$ and by $\tilde{W}_T^{1,p}(\tilde{X}) = \{u \in W_T^{1,p}(\tilde{X}) : \int_0^T u(t)\, dt = \theta_{\tilde{X}}\}$. The norm over $W_T^{1,p}(\tilde{X})$ is defined by

$$\|u\|_{W_T^{1,p}} = \left(\int_0^T \|u(t)\|_{\tilde{X}}^p \, dt + \int_0^T \|\dot{u}(t)\|_{\tilde{X}}^p \, dt \right)^{\frac{1}{p}}$$

and

$$\|u\|_{L^p} = \left(\int_0^T \|u(t)\|_{\tilde{X}}^p \, dt \right)^{\frac{1}{p}}, \qquad \|u\|_\infty = \max_{t \in [0,T]} \|u(t)\|_{\tilde{X}}.$$

It is necessary to recall the infinite dimensional variants of some very well known inequalities (for proofs and details see Dincă and Paşca):

(a) If $u \in W_T^{1,p}(\tilde{X})$ then $\|u\|_\infty \leqslant (T^{-\frac{1}{p}} + T^{\frac{1}{q}}) \|u\|_{W_T^{1,p}}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

(b) If $u \in \tilde{W}_T^{1,p}(\tilde{X})$ then $\|u\|_\infty \leqslant T^{\frac{1}{q}} \|\dot{u}\|_{L^p}$ (Sobolev inequality).

3.1. FENCHEL TRANSFORM AND THE DIFFERENTIABILITY OF CONVEX FUNCTIONALS

We recall the basic tool about the Fenchel transform and the differentiability of convex functionals. Let X be a real Banach space and X^* its dual.

DEFINITION 1. *Let* $F : X \to (-\infty, \infty]$ *be a proper functional. Then* $F^* : X^* \to (-\infty, \infty]$ *given by*

$$F^*(x^*) = \sup_{x\in X} \{\langle x^*, x\rangle_{X^*,X} - F(x)\}, \quad \forall x^* \in X^*$$

is called the Fenchel transform *of* F.

DEFINITION 2. *A functional* $F : X \to (-\infty, \infty]$ *is called* uniformly convex *if*

$$\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \geqslant \gamma^2 \lambda(1-\lambda)\|x-y\|_X^2, \tag{10}$$

$\forall x, y \in dom\, F$, $\forall \lambda \in [0, 1]$ *and there* $\gamma^2 > 0$.

DEFINITION 3. *Given the proper convex functional* $F : X \to (-\infty, \infty]$, *the* subdifferential *of such a function is the mapping* $\partial F : X \to 2^{X^*}$ *given by*

$$\partial F(x) = \begin{cases} \emptyset; & x \notin dom\, F \\ \{x^* \in X^* : F(y) - F(x) \geqslant \langle x^*, y - x\rangle_{X^*,X},\ \forall y \in X\}; & x \in dom\, F. \end{cases}$$

PROPOSITION 1. *If* $F : X \to (-\infty, \infty]$ *is proper and convex functional, then it follows:*

1) $x^* \in \partial F(x) \Leftrightarrow F(x) + F(x^*) = \langle x^*, x\rangle_{X^*,X}$
2) $x^* \in \partial F(x) \Rightarrow \pi x \in \partial F^*(x^*)$
3) *Moreover, if* F *is lower semicontinuous then* $[x^* \in \partial F(x) \Leftrightarrow \pi x \in \partial F^*(x^*)]$. *(We denoted with* π *the canonical injection of* X *in* X^{**}.*)*

PROPOSITION 2. *Let* $F : X \to (-\infty, \infty]$ *be a convex, proper and lower semicontinuous such that*

$$-\beta \leqslant F(x) \leqslant \frac{\alpha}{q}\|x\|_X^q + \gamma$$

with $\alpha > 0$, $q > 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$. *Then, if* $x^* \in \partial F(x)$ *it follows:*

$$\alpha^{-\frac{p}{q}} \frac{1}{p}\|x^*\|_{X^*}^p \leqslant \langle x^*, x\rangle_{X^*,X} + \beta + \gamma$$

and

$$\|x^*\|_{X^*} \leqslant \{p\alpha^{\frac{p}{q}}(\|x\|_X + \beta + \gamma) + 1\}^{q-1}.$$

PROPOSITION 3. *Let* X *be a reflexive Banach space,* $F : X \to (-\infty, \infty]$ *is proper,* *lower semicontinuous and uniformly convex such that* $\frac{F(x)}{\|x\|_X} \to \infty$ *when* $\|x\|_X \to \infty$. *Then* $F^* \in C^1(X^*, \mathbf{R})$ *and* F^* *is bounded to below.*

PROPOSITION 4. *If* $F : X \to \mathbf{R}$ *is convex and differentiable at* x, *then* $\partial F(x) = \{\nabla F(x)\}$.

The proofs of the following results is given in Dincă and Paşca. When X is finite dimensional the proofs was given in Mawhin and Willem (1989).

THEOREM 2. *Let* X *be a Banach space. Let* $L : [0, T] \times X \to \mathbf{R}$, $(t, x) \mapsto L(t, x)$ *be measurable in* t *for each* $x \in X$ *and continuously differentiable in* x *for almost every* $t \in [0, T]$. *If there exists* $a \in C(\mathbf{R}_+, \mathbf{R}_+)$, $b \in L^1(0, T; \mathbf{R}_+)$ *such that, for a.e.* $t \in [0, T]$ *and every* $x \in X$, *one has*

$$|L(t, x)| \leq a(\|x\|_X)b(t)$$
$$\|D_x L(t, x)\| \leq a(\|x\|_X)b(t)$$

then the functional $\varphi : H_T^1(X) \to \mathbf{R}$ *defined by* $\varphi(u) = \int_0^T L(t, u(t))\, dt$ *is differentiable on* $H_T^1(X)$ *and*

$$\langle \varphi'(u), v \rangle = \int_0^T \langle D_x L(t, u(t)), v(t) \rangle_{X^*, X}\, dt .$$

THEOREM 3. *Let* X *be a Hilbert space,* $\tilde{X} = X \times X$, *and* $H : [0, T] \times \tilde{X} \to \mathbf{R}$ *such that* $(t, u) \mapsto H(t, u)$ *be measurable in* t *for each* $u \in \tilde{X}$, *uniformly convex and continuously differentiable in* u *for almost every* $t \in [0, T]$. *Let* $A : D(A) \subset X \to X$ *an operator like in Introduction. Assume that there exists* $\alpha > 0$, $\delta > 0$, β, $\gamma \in L^2(0, T; \mathbf{R}_+)$, *such that, for all* $u \in \tilde{X}$ *and a.e.* $t \in [0, T]$, *one has*

$$\delta\, \frac{\|u\|_{\tilde{X}}^2}{2} - \beta(t) \leq H(t, u) \leq \alpha\, \frac{\|u\|_{\tilde{X}}^2}{2} + \gamma(t) .$$

Then, the dual action $\varphi(v) = \int_0^T [H^*(t, v(t)) - \frac{1}{2}\langle \mathscr{A}^{-1} v(t), v(t) \rangle_{\tilde{X}}]\, dt$ *is differentiable on* $R(\tilde{\mathscr{A}})$ *and, if* $v \in R(\tilde{\mathscr{A}})$ *is a critical point of* φ, *function* u *defined by* $u(t) = \nabla H^*(t, v(t))$ *satisfies*

$$\mathscr{A} u(t) = \nabla H(t, u(t)) , \quad u(0) = u(T) .$$

3.2. THE SZULKIN'S RESULT

In Szulkin (1986), the author introduces the following framework:

Let X be a real Banach space and I a function on X satisfying the following hypothesis:

(H) $I = \Phi + \psi$, *where* $\Phi \in C^1(X, \mathbf{R})$ *and* $\psi : X \to (-\infty, \infty]$ *is convex, proper and lower semicontinuous.*

DEFINITION 4. *A point $u \in X$ is said to be a* critical point *of I if $u \in dom\ \psi$ and if it satisfies the inequality*

$$\langle \Phi'(u), v - u\rangle_{X^*,X} + \psi(v) - \psi(u) \geqslant 0\,, \quad \forall v \in X\,.$$

A number $c \in \mathbf{R}$ such that $I^{-1}(c)$ contains a critical point will be called a critical value.

DEFINITION 5. *We say that I satisfies the* Palais-Smale condition (*in short* (*PS*)) *if each sequence (u_n) in X such that $I\,(u_n) \to c \in \mathbf{R}$ and*

$$\langle \Phi'(u_n), v - u_n\rangle_{X^*,X} + \psi(v) - \psi(u_n) \geqslant \langle z_n, v - u_n\rangle_{X^*,X} \quad \forall v \in X\,,$$

where $z_n \to \theta_{X^*}$, possesses a convergent subsequence.

Under the conditions described above, Szulkin prove (see Szulkin (1986), Theorem 3.2):

THEOREM 4 (Mountain Pass Theorem). *Suppose that $I : X \to (-\infty, \infty]$ is a function satisfying (H), (PS) and*

i) $I(\theta_X) = 0$ and there exist δ, $\rho > 0$ such that $I_{|\partial B_\rho} \geqslant \delta$,

ii) $I(e) \leqslant 0$ for some $e \not\in \overline{B}_\rho$.

Then I has a critical value $c \geqslant \delta$ which may be characterized by

$$c = \inf_{f \in \Gamma} \sup_{t \in [0,1]} I(f(t))\,,$$

where $\Gamma = \{f \in C([0,1], X) : f(0) = \theta_X,\ f(1) = e\}$.

4. Proof of Theorem 1

The proof of Theorem 1 will be a combination of the dual least action principle and the mountain pass theorem, and requires several preliminary lemmas. We will follow the ideas from finite dimensional case studied in Mawhin and Willem (1989).

LEMMA 1. *If*

$$M = \sup_{\|u\|_{\tilde{X}}=1} H(u)\,, \qquad m = \inf_{\|u\|_{\tilde{X}}=1} H(u) \tag{11}$$

then

$$\|u\|_{\tilde{X}} \leqslant 1 \Rightarrow H(u) \leqslant M\|u\|^2_{\tilde{X}}\,, \qquad \|u\|_{\tilde{X}} \geqslant 1 \Rightarrow H(u) \geqslant m\|u\|^2_{\tilde{H}}\,. \tag{12}$$

Proof. First of all, we will see that $M < \infty$ and $m > 0$. From (8) follow that for $\|u\|_{\tilde{X}} = 1$ we have $H(u) \leqslant \alpha$ and therefore $M = \sup_{\|u\|_{\tilde{X}}=1} H(u) \leqslant \alpha < \infty$. Now we will prove that $m > 0$.

From Definition 2 we have

$$\lambda H(u) + (1-\lambda)H(v) - H(\lambda u + (1-\lambda)v) \geqslant \lambda(1-\lambda)\gamma^2 \|u-v\|^2_{\tilde{X}}$$

$\forall u,\ v \in \tilde{X},\ \forall \lambda \in (0,1)$. For $v = \theta_{\tilde{X}}$ and $u \in \tilde{X}$ with $\|u\|_{\tilde{X}} = 1$, because $H(\theta_{\tilde{X}}) = 0$ follow

$$H(u) - \frac{H(\lambda u)}{\lambda} \geqslant (1-\lambda)\gamma^2 . \tag{13}$$

Since $H \in C^1(\tilde{X}, \mathbf{R})$ and $\nabla H(\theta_{\tilde{X}}) = \theta_{\tilde{X}}$, we have

$$\lim_{\|h\| \to 0} \frac{H(v+h) - H(v) - \langle \nabla H(v), h\rangle}{\|h\|} = 0 . \tag{14}$$

If we take in (14) $h = \lambda u$, $\|u\|_{\tilde{X}} = 1$, $\lambda > 0$, $\lambda \to 0$, $v = \theta_{\tilde{X}}$ we obtain

$$\lim_{\lambda \to 0, \lambda > 0} \frac{H(\lambda u)}{\lambda} = 0 .$$

Therefore, from (13) $H(u) \geqslant \gamma^2 > 0$, $\forall u$, $\|u\|_{\tilde{X}} = 1$ which implies $\inf_{\|u\|_{\tilde{X}}=1} H(u) = m > 0$.

For will prove (12), we define $f : \mathbf{R} \to \mathbf{R}$, $f(s) = H(sv)$ for some fixed $v \in \tilde{X}$. Assumption (7) implies that $sf'(s) \geqslant 2f(s)$. Thus, if $s \geqslant 1$, $f(s) \geqslant s^2 f(1)$ i.e. $H(sv) \geqslant s^2 H(v)$. If $\|u\|_{\tilde{X}} \leqslant 1$ this implies

$$H\left(\frac{u}{\|u\|_{\tilde{X}}}\right) \geqslant \|u\|^{-2}_{\tilde{X}} H(u)$$

and if $\|u\|_{\tilde{X}} \geqslant 1$, this implies

$$H(u) = H\left(\|u\|_{\tilde{X}} \frac{u}{\|u\|_{\tilde{X}}}\right) \geqslant \|u\|^2_{\tilde{X}} H\left(\frac{u}{\|u\|_{\tilde{X}}}\right) . \qquad \square$$

LEMMA 2. *The function H^* is continuously differentiable on $\tilde{X}$ and, if*

$$m^* = \inf_{\|v\|_{\tilde{X}}=1} H^*(v) , \quad M^* = \sup_{\|v\|_{\tilde{X}}=1} H^*(v) , \quad \alpha^* = \frac{1}{4\alpha}$$

we have $m^* > 0$ and

$$2H^*(v) \geqslant \langle \nabla H^*(v), v\rangle \tag{15}$$

$$\|v\|_{\tilde{X}} \leqslant 1 \Rightarrow H^*(v) \geqslant m^* \|v\|^2_{\tilde{X}} \tag{16}$$

$$\|v\|_{\tilde{X}} \geqslant 1 \Rightarrow H^*(v) \leqslant M^* \|v\|^2_{\tilde{X}} \tag{17}$$

$$H^*(v) \geqslant \alpha^* \|v\|^2_{\tilde{X}} \tag{18}$$

for all $v \in \tilde{X}$.

Proof. First of all, we will prove that $m^* > 0$ and $M^* < \infty$. Relation (8) implies relation (18), and therefore $m^* > 0$.

By contradiction we suppose that $M^* = \sup_{\|v\|_{\tilde{X}}=1} H^*(v) = \infty$. It follows that there exists a sequence (v_n), $\|v_n\|_{\tilde{X}} = 1$ such that $H^*(v_n) \to \infty$. Since H is uniformly convex and, by (12), such that $\frac{H(u)}{\|u\|_{\tilde{X}}} \to \infty$ as $\|u\|_{\tilde{X}} \to \infty$, Proposition 3 implies that $H^* \in C^1(\tilde{X}, \mathbf{R})$. Now it follows from Proposition 1 and Proposition 4 that for fixed $v \in \tilde{X}$ exists a unique $u \in \tilde{X}$ such that

$$v = \nabla H(u) \Leftrightarrow u = \nabla H^*(v) \Leftrightarrow H^*(v) = \langle v, u \rangle - H(u)\,. \tag{19}$$

Therefore exist a sequence $(u_n) \in \tilde{X}$ such that

$$H^*(v_n) = \langle v_n, u_n \rangle - H(u_n)\,, \quad \|v_n\|_{\tilde{X}} = 1\,.$$

Follows that

$$H^*(v_n) \leqslant \|u_n\|_{\tilde{X}} - H(u_n)$$

and therefore

$$\|u_n\|_{\tilde{X}} - H(u_n) \to \infty\,. \tag{20}$$

From uniformly convexity of H, for $\lambda = \frac{1}{2}$, $v = \theta_{\tilde{X}}$ we have

$$H(u_n) \geqslant \frac{1}{2}\gamma^2 \|u_n\|^2_{\tilde{H}} + 2H\left(\frac{u_n}{2}\right)$$

which implies

$$\|u_n\|_{\tilde{X}} - H(u_n) \leqslant \|u_n\|_{\tilde{X}} - \frac{1}{2}\gamma^2 \|u_n\|^2_{\tilde{X}} - 2H\left(\frac{u_n}{2}\right). \tag{21}$$

From (20) and (21) result

$$H\left(\frac{u_n}{2}\right) \to -\infty\,. \tag{22}$$

Relation (22) is in contradiction with the fact that in our assumptions, the Hamiltonian H is bounded to below.

From (19), assumption (7) implies that

$$H^*(v) = \langle v, u \rangle - H(u) \geqslant \left(1 - \frac{1}{2}\right)\langle v, u \rangle = \frac{1}{2}\langle v, \nabla H^*(v)\rangle\,.$$

Like in the proof of Lemma 1, (15) implies (16) and (17). □

REMARK 1. We know that the function H^* is convex but we do not know that H^* is uniformly convex. In the proof of Lemma 2 it is not necessary that H^* be uniformly convex.

REMARK 2. If we denote $\sup_{\|v\|_{\tilde{X}} \leqslant 1} H^*(v) = M_1^*$ and proceed like in the proof of Lemma 2 it follows that $M_1^* < \infty$. Since, by (17), $H^*(v) \leqslant \max(M^* + M_1^*)(1 + \|v\|^2_{\tilde{X}})$ for all $v \in \tilde{X}$. Proposition 2 and Theorem 2 imply that the dual action defined by

$$\varphi(v) = \int_0^T \left[H^*(v(t)) - \frac{1}{2}\langle \mathscr{A}^{-1} v(t), v(t)\rangle_{\tilde{X}} \right] dt$$

is differentiable on $R(\tilde{\mathscr{A}})$.

LEMMA 3. *There exists* $C > 0$ *such that, for each* $v \in R(\tilde{\mathscr{A}})$ *one has*

$$\int_0^T \langle \mathscr{A}^{-1} v(t), v(t)\rangle_{\tilde{X}}\, dt \leqslant C\|v\|^2_{L^2(0,T;\tilde{X})} .$$

Proof. Obviously

$$\int_0^T \langle \mathscr{A}^{-1} v(t), v(t)\rangle_{\tilde{X}}\, dt \leqslant \int_0^T \|\mathscr{A}^{-1}\|\, \|v\|^2_{\tilde{X}}\, dt = C\|v\|^2_{L^2(0,T;\tilde{X})} . \qquad \square$$

LEMMA 4. *Every sequence* (v_j) *in* $R(\tilde{\mathscr{A}}) \subset L^2(0, T; \tilde{X})$ *such that* $(\varphi(v_j))$ *is bounded and* $\varphi'(v_j) \to \theta_{(L^2(0,T;\tilde{X}))^*}$ *contains a convergent subsequence.*

Proof. Theorem 2 imply that the dual action φ is differentiable and

$$\langle \varphi'(v), w\rangle = \langle \nabla H^*(v(\cdot)) - \tilde{\mathscr{A}}^{-1} v, w\rangle_{L^2(0,T;\tilde{X})} .$$

The Riesz representation theorem imply the existence of a sequence (f_j) in $L^2(0, T; \tilde{X})$ such that $\|f_j\|_{L^2} \to 0$ as $j \to \infty$ and

$$\nabla H^*(v_j(\cdot)) - \tilde{\mathscr{A}}^{-1} v_j = f_j , \quad \langle f_j, w\rangle_{L^2(0,T;\tilde{X})} \to \theta_{(L^2(0,T;\tilde{X}))^*} \tag{23}$$

for all $w \in R(\tilde{\mathscr{A}})$. Using Lemma 3, (8) and (18) we obtain

$$\varphi(v_j) = \int_0^T H^*(v_j(t))\, dt - \frac{1}{2}\int_0^T \langle \mathscr{A}^{-1} v_j(t), v_j(t)\rangle_{\tilde{X}}\, dt$$
$$\geqslant \frac{1}{2}\left(\frac{1}{2\alpha} - \|\mathscr{A}^{-1}\|\right)\|v_j\|^2_{L^2(0,T;\tilde{X})} .$$

Because $(\varphi(v_j))$ is bounded it follows (v_j) is bounded in $L^2(0, T; \tilde{X})$. Going if necessary to a subsequence, we can assume because $\tilde{\mathscr{A}}^{-1}$ is compact, that $v_j \rightharpoonup v$ in $L^2(0, T; \tilde{X})$, $\tilde{\mathscr{A}}^{-1} v_j \to w$ in $L^2(0, T; \tilde{X})$. But $\tilde{\mathscr{A}}^{-1}$ is selfadjoint, therefore weakly closed, and we have $w = \tilde{\mathscr{A}}^{-1} v$. From (23) we have

$$\nabla H^*(v_j(\cdot)) = \tilde{\mathscr{A}}^{-1} v_j + f_j \to \tilde{\mathscr{A}}^{-1} v ,$$

and by duality

$$v_j = \nabla H(\tilde{\mathscr{A}}^{-1} v_j(\cdot) + f_j(\cdot)) . \tag{24}$$

Now, assumption (8) and Proposition 2 imply that ∇H maps continuously $L^2(0, T; \tilde{X})$ into $L^2(0, T; \tilde{X})$, so that

$$v_j = \nabla H(\tilde{\mathscr{A}}^{-1} v_j(\cdot) + f_j(\cdot)) \to \nabla H(\tilde{\mathscr{A}}^{-1} v) . \qquad \square$$

Proof of Theorem 1

1) Now we can write the dual action on the following form:

$$\varphi = \varphi_1 + \varphi_2\,, \quad \text{where}$$

$$\varphi_1(v) = -\frac{1}{2}\int_0^T \langle \mathscr{A}^{-1}v(t), v(t)\rangle_{\tilde{X}}\, dt, \quad \varphi_2(v) = \int_0^T H^*(v(t))\, dt$$

with φ_1 differentiable and φ_2 convex and differentiable (see Remark 2). By Theorem 2 it follows

$$\langle \varphi_1'(v), w\rangle = -\int_0^T \langle \mathscr{A}^{-1}v(t), w(t)\rangle\, dt\,.$$

By Hölder's inequality it follows $|\langle \varphi_1'(v), w\rangle| \leqslant \|\mathscr{A}^{-1}\|\,\|v\|_{L^2}\|w\|_{L^2}$ which implies the continuity of φ_1' and hence φ satisfies (H). Using Proposition 4, it is obvious that, for φ, the definition of critical point (Definition 4) as well as the (PS) condition (Definition 5) coincide with the usual ones. We shall apply Theorem 4 to φ. By Lemma 4, φ satisfies the (PS) condition for every $c \in \mathbf{R}$.

2) Since

$$\varphi(v) \geqslant \int_0^T \left[H^*(v(t)) - \frac{1}{2}\langle \mathscr{A}^{-1}v(t), v(t)\rangle_{\tilde{X}} \right],$$

$$H^*(v) \geqslant \frac{1}{4\alpha}\|v\|_{\tilde{X}}^2 \quad \text{and} \quad \|\mathscr{A}^{-1}\| < \frac{1}{2\alpha}$$

we obtain

$$\varphi(v) > \frac{1}{2}\left(\frac{1}{2\alpha} - \|\mathscr{A}^{-1}\|\right)\|v\|^2_{L^2(0,T;\tilde{X})} > \chi(\theta_{L^2(0,T;\tilde{X})}) = 0$$

$$\text{if} \quad 0 < \|v\|_{L^2(0,T;\tilde{X})} < \rho$$

$$\varphi(v) > \delta > 0 \quad \text{if} \quad \|v\|_{L^2(0,T;\tilde{X})} = \rho\,.$$

3) Let

$$v_k(t) = \mathscr{A}w_k(t) \quad \text{where} \quad w_k(t) = \left(\cos\frac{2k\pi}{T}\,t\right)\tilde{c} - \left(\sin\frac{2k\pi}{T}\,t\right)J\tilde{c}$$

and $k \in \mathbf{Z}$, $\tilde{c} = (c, c)$, $c \in D(A)$, $\|c\|_X = \frac{1}{\sqrt{2}}$. Then, $\|w_k\|_{L^2(0,T;\tilde{X})} = \sqrt{T}$ for all $k \in \mathbf{Z}$ and

$$\varphi(v_k) = \int_0^T H^*(v_k(t))\, dt + k\pi\,.$$

By

$$H^*(v) \leqslant \max(M^*, M_1^*)(1 + \|v\|^2)\,,$$

we obtain

$$\varphi(v_k) \leqslant \max(M^*, M_1^*)T(1 + \|\mathscr{A}\|) + k\pi\,.$$

Obviously we can choose $k \in \mathbf{Z}$ such that $\varphi(v_k) < 0$ and such that $\|v_k\|_{L^2(0,T;\tilde{X})} > \rho$.

4) Theorem 4 implies the existence of a critical point v of φ such that $\varphi(v) > \varphi(\theta_{\tilde{X}})$. By Theorem 3,

$$u(t) = \nabla H^*(v(t))$$

is a nontrivial T-periodic solution of (9). □

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